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Asymptotic existence theorems for frames and group divisible designs

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Abstract

In this paper, we establish an asymptotic existence theorem for group divisible designs of type m^n with block sizes in any given set K of integers greater than 1. As consequences, we will prove an asymptotic existence theorem for frames and derive a partial asymptotic existence theorem for resolvable group divisible designs.

Keywords: Block designs; Frames; Group divisible designs

1. Introduction

We refer to [1] for basic concepts in combinatorial designs. Here we give a few additional concepts that we need throughout the paper.

Definition 1.1. Let v, λ be positive integers and let K be a set of positive integers. A *group divisible design* (or a GDD for short) of order v is a triple $(X, \mathcal{G}, \mathcal{B})$, where

(1) X is a set of v elements,
(2) $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a set of subsets of X which partition X (called groups),

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(3) $\mathcal{B}$ is a family of subsets of X each of cardinality from K (called blocks),
(4) every pair of elements from X is in exactly λ blocks if they are from different groups, 0 blocks if they are in the same group.

If all groups $G_1, G_2, \ldots, G_n$ have the same size m, such a group divisible design is said to be of type m^n, and for convenience, we denote such a group divisible design by a (K, λ)-GDD of type m^n, or a K-GDD of type m^n whenever $\lambda = 1$. If $K = \{k\}$, then all blocks have the same size k. Clearly, an (n, k, λ)-design (or BIBD) is a special group divisible design $(\{k\}, \lambda)$-GDD of type 1^n. We say a design is *resolvable* if its blocks can be partitioned into parallel classes such that every element occurs in each class exactly once, i.e., each parallel class partitions X. For example, a Kirkman triple system of order v is a resolvable $(v, 3, 1)$-design. We will denote a resolvable (K, λ)-GDD of type m^n by a (K, λ)-RGDD of type m^n, or a K-RGDD of type m^n whenever $\lambda = 1$.

Frames defined in the following form another kind of very useful combinatorial structures (for more on frames, see [4] and [10]).

Definition 1.2. Let X be a set of v elements and $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of X. Let $\lambda \geqslant 1$ and K be a set of positive integers. A (K, λ)-*frame* is a group divisible design $(X, \mathcal{G}, \mathcal{B})$ whose blocks are partitioned into partial parallel classes so that each partial parallel class partitions $X - G_i$, for some $G_i \in \mathcal{G}$.

If all $G_1, G_2, \ldots, G_n$ in a frame have the same size m, such a frame is said to be of type m^n. We simply use K-frame of type m^n to denote such a frame when $\lambda = 1$. For example, if we delete a vertex x and all blocks containing x from a Kirkman triple system of order v (i.e., a resolvable $(v, 3, 1)$-design), we obtain a $\{3\}$-frame of type $2^{\frac{v-1}{2}}$.

Constructing (or studying existence problems of) various kinds of designs is one of central tasks in design theory. Though a lot of progresses have been made, the spectrum for the existence of each kind of designs is far from being completely settled. In 1973, R.M. Wilson [11,12], and Ray-Chaudhuri and R.M. Wilson [8] proved the following asymptotic existence theorems.

Theorem 1.3. (R.M. Wilson [12]) *Given fixed integers $k \geqslant 2$ and $\lambda \geqslant 1$, there exists v_0 such that (v, k, λ)-designs exist for all integers $v \geqslant v_0$ that satisfy the necessary conditions $\lambda(v - 1) \equiv 0 \pmod{k - 1}$ and $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$.*

Theorem 1.4. (Ray-Chaudhuri and R.M. Wilson [8]) *Given a fixed integer $k \geqslant 2$, there exists v_0 such that resolvable $(v, k, 1)$-designs exist for all integers $v \geqslant v_0$ that satisfy the necessary conditions $(v - 1) \equiv 0 \pmod{k - 1}$ and $v \equiv 0 \pmod{k}$.*

Then, in 1984, Theorem 1.4 was extended to resolvable (v, k, λ)-designs for $\lambda > 1$ by J.X. Lu [6].

Theorem 1.5. (J.X. Lu [6]) *Given fixed integers $k \geqslant 2$ and $\lambda \geqslant 1$, there exists v_0 such that resolvable (v, k, λ)-designs exist for all integers $v \geqslant v_0$ that satisfy the necessary conditions $\lambda(v - 1) \equiv 0 \pmod{k - 1}$ and $v \equiv 0 \pmod{k}$.*

In his thesis [2], K.I. Chang proved the following asymptotic existence result for group divisible designs where all blocks have the same size k. A different proof for this result was given by E.R. Lamken and R.M. Wilson [5].

Theorem 1.6. (K.I. Chang [2]) *Given fixed integers $k \geqslant 2$, $\lambda \geqslant 1$, and $m \geqslant 1$, there exists n_0 such that a $(\{k\}, \lambda)$-GDD of type m^n exists for all integers $n \geqslant n_0$ that satisfy the necessary conditions $\lambda m(n-1) \equiv 0 \pmod{k-1}$ and $\lambda m^2 n(n-1) \equiv 0 \pmod{k(k-1)}$.*

In 2002, H. Mohacsy and D.R. Ray-Chaudhuri [7] proved a partial asymptotic existence result for group divisible designs with fixed number of groups.

Theorem 1.7. (H. Mohacsy and D.R. Ray-Chaudhuri [7]) *Let k and n be fixed integers satisfying $2 \leqslant k \leqslant n$. Then there exists an integer m_0 such that a $\{k\}$-GDD of type m^n exists for all integers $m \geqslant m_0$ if the conditions $(n-1) \equiv 0 \pmod{k-1}$ and $n(n-1) \equiv 0 \pmod{k(k-1)}$ are satisfied.*

Note that both Theorems 1.6 and 1.7 deal with group divisible designs whose blocks have the same size k. In this paper, we extend Theorem 1.6 to the following asymptotic existence theorem for (K, λ)-GDDs of type m^n, where the sizes of blocks form any given set K of integers greater than 1.

Given a set K of integers greater than 1, let $\alpha(K)$ be the greatest common divisor of the integers in $\{k-1\colon\ k \in K\}$ and let $\beta(K)$ be the greatest common divisor of the integers in $\{k(k-1)\colon\ k \in K\}$.

Theorem 1.8. *Given fixed integers $\lambda \geqslant 1$ and $m \geqslant 1$, and a fixed set K of integers greater than 1, there exists n_0 such that a (K, λ)-GDD of type m^n exists for all integers $n \geqslant n_0$ that satisfy the necessary conditions*

$$\lambda m(n-1) \equiv 0 \big(\mathrm{mod}\ \alpha(K)\big) \quad \textit{and} \quad \lambda m^2 n(n-1) \equiv 0 \big(\mathrm{mod}\ \beta(K)\big).$$

As a consequence to Theorem 1.8, we establish the following asymptotic existence theorem for frames.

Theorem 1.9. *Given any integers $k \geqslant 2$ and $g \geqslant 1$, there exists u_0 such that all $\{k\}$-frames of type g^u exist for all $u \geqslant u_0$ satisfying the necessary conditions $g \equiv 0 \pmod{k-1}$ and $g(u-1) \equiv 0 \pmod{k}$.*

By using Theorem 1.9, we will derive a partial asymptotic existence result for resolvable group divisible designs in Section 4.

2. Proof of Theorem 1.8

To prove Theorem 1.8, we need to use a powerful theorem by E.R. Lamken and R.M. Wilson in [5]. Before stating the theorem, we first introduce certain necessary concepts and notations from [5].

Let $K_n^{(r,\lambda)}$ be a complete digraph on n vertices with exactly λ edges of color i joining any vertex x to any vertex y for every color i in a set of r colors.

A family $\mathcal{F}$ of subgraphs of $K_n^{(r,\lambda)}$ will be called a *decomposition* of $K_n^{(r,\lambda)}$ if every edge $e \in E(K_n^{(r,\lambda)})$ belongs to exactly one member in $\mathcal{F}$. Given a family Φ of edge-r-colored digraphs, a Φ-*decomposition* of $K_n^{(r,\lambda)}$ is a decomposition $\mathcal{F}$ such that every graph $F \in \mathcal{F}$ is isomorphic to some graph $G \in \Phi$.

For a vertex x of an edge-r-colored digraph G, the *degree-vector* of x is the $2r$-vector

$$\mathbf{d}(x) = \bigl(\mathrm{in}_1(x), \mathrm{out}_1(x), \mathrm{in}_2(x), \mathrm{out}_2(x), \ldots, \mathrm{in}_r(x), \mathrm{out}_r(x)\bigr),$$

where $\mathrm{in}_j(x)$ and $\mathrm{out}_j(x)$ denote, respectively, the indegree and outdegree of vertex x in the spanning subgraph of G by edges of color j, $1 \leqslant j \leqslant r$. We denote by $\alpha(G)$ the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \ldots, t)$ is an integral linear combination of the vectors $\mathbf{d}(x)$ as x ranges over the vertex set $V(G)$ of G. Equivalently, $\alpha(G)$ is the least positive integer t_0 such that $(t_0, t_0, \ldots, t_0)$ is an integral linear combination of the vectors $\mathbf{d}(x)$.

Let Φ be a family of simple edge-r-colored digraphs and let $\alpha(\Phi)$ denote the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \ldots, t)$ is an integral linear combination of the vectors $\mathbf{d}(x)$ as x ranges over all vertices of all graphs in Φ. For each graph $G \in \Phi$, let $\mu(G) = (m_1, m_2, \ldots, m_r)$, where m_i is the number of edges of color i in G. We denote by $\beta(\Phi)$ the greatest common divisor of the integers m such that $(m, m, \ldots, m)$ is an integral linear combination of the vectors $\mu(G)$, $G \in \Phi$. Equivalently, $\beta(\Phi)$ is the least positive integer m_0 such that $(m_0, m_0, \ldots, m_0)$ is an integral linear combination of the vectors $\mu(G)$.

A graph $G_0 \in \Phi$ is *useless* when it cannot occur in any Φ-*decomposition* of $K_n^{(r,\lambda)}$. We say that Φ is *admissible* when no member of Φ is useless. Equivalently, Φ is admissible if and only if there exists a positive rational linear relation

$$(1, 1, \ldots, 1) = \sum_{G \in \Phi} c_G \mu(G) \quad \text{with all } c_G > 0.$$

Here is the powerful result which is Corollary 13.3 (or Theorem 1.2 when $\lambda = 1$) in [5].

Theorem 2.1. (E.R. Lamken and R.M. Wilson [5]) *Let Φ be an admissible family of simple edge-r-colored digraphs. Then there exists a constant $n_0 = n_0(\Phi)$ such that Φ-decompositions of $K_n^{(r,\lambda)}$ exist for all $n \geqslant n_0$ satisfying the congruences*

$$\lambda(n-1) \equiv 0 \pmod{\alpha(\Phi)},$$
$$\lambda n(n-1) \equiv 0 \pmod{\beta(\Phi)}.$$

It is shown by E.R. Lamken and R.M. Wilson in [5] that the existence of certain combinatorial structures can be seen to be equivalent to the existence of a Φ-decomposition of $K_n^{(r,\lambda)}$ for some Φ, r, and λ. To establish such an equivalence for a given combinatorial structure, it usually involves two steps: First, find appropriate Φ, r, and λ; and then we need to show that the necessary conditions for the combinatorial structure imply an integer n satisfying the two congruences in Theorem 2.1. From the definitions for $\alpha(\Phi)$ and $\beta(\Phi)$, it is easy to see that $\lambda n(n-1) \equiv 0 \pmod{\beta(\Phi)}$ is equivalent to showing that the vector $\lambda n(n-1)(1, 1, \ldots, 1)$ is an integral linear combination of the vectors $\mu(G)$ over all $G \in \Phi$, and $\lambda(n-1) \equiv 0 \pmod{\alpha(\Phi)}$ is equivalent to showing that the vector $\lambda(n-1)(1, 1, \ldots, 1)$ is an integral linear combination of the vectors $\mathbf{d}(x)$, as x ranges over all vertices of digraphs $G \in \Phi$. This can be done by applying the following well-known lemma from [9].

Lemma 2.2. *Let M be a rational s by t matrix and* $\mathbf{c}$ *a rational column vector of length s. The equation $M\mathbf{x}=\mathbf{c}$ has an integral solution* $\mathbf{x}$*, a column vector of length t, if and only if*

$\mathbf{y}M$ *integral implies* $\mathbf{yc}$ *is an integer*

for all rational row vectors $\mathbf{y}$ *of length s.*

The following proof for Theorem 1.8 is motivated by the method used for the proof of Theorem 8.1 in [5].

Proof of Theorem 1.8. It is easy to see that the conditions in Theorem 1.8 are necessary for the existence of such a group divisible design.

Given a set K of integers greater than 1, we will show that the existence of a (K,λ)-GDD of type m^n is equivalent to the existence of a Φ-decomposition of $K_n^{(r,\lambda)}$, where $r=m^2$ and Φ is the family of edge-r-colored graphs described below.

As colors, we use the ordered pairs from $\{1,2,\ldots,m\}$. For each $k\in K$, let $\mathcal{T}(m,k)$ denote the set of m-sequences $\mathbf{t}=(t_1,t_2,\ldots,t_m)$ of nonnegative integers summing to k, let $G(\mathbf{t},k)$ be the simple digraph with vertex set $V(G(\mathbf{t},k))=T_1\cup T_2\cup\cdots\cup T_m$ where $T_1,T_2,\ldots,T_m$ are disjoint with $|T_i|=t_i$ and for all distinct $x,y\in V(G(\mathbf{t},k))$, there is exactly one edge from x to y of color (i,j) where i,j are such that $x\in T_i$ and $y\in T_j$ (the digraph $G(\mathbf{t},k)$ is simple because $T_1,T_2,\ldots,T_m$ are disjoint and there is only one directed edge (x,y) of color (i,j) between every pair of distinct vertices x and y, where i,j are such that $x\in T_i$ and $y\in T_j$). Let Φ be the collection of all such $G(\mathbf{t},k)$ for all $\mathbf{t}\in\mathcal{T}(m,k)$ and all $k\in K$.

To obtain a (K,λ)-GDD of type m^n from a Φ-decomposition $\mathcal{F}$ of $K_n^{(r,\lambda)}$, if it exists, let $V=V(K_n^{(r,\lambda)})$ and let $X=V\times\{1,2,\ldots,m\}$. Set $\mathcal{G}=\{\{x\}\times\{1,2,\ldots,m\}\colon\ x\in V\}$. For each $F\in\mathcal{F}$, there is a unique partition $V(F)=S_1\cup S_2\cup\cdots\cup S_m$ so that the edge from x to y in F has color (i,j) if and only if $x\in S_i$ and $y\in S_j$. Let

$$B_F=\bigcup_{i=1}^{m}S_i\times\{i\}$$

and let $\mathcal{B}=\{B_F\colon\ F\in\mathcal{F}\}$. Then it is not difficult to check that $(X,\mathcal{G},\mathcal{B})$ is a (K,λ)-GDD of type m^n.

To apply Theorem 2.1 to obtain a Φ-decomposition $\mathcal{F}$ of $K_n^{(r,\lambda)}$, we need to show that $\lambda m(n-1)\equiv 0 \pmod{\alpha(K)}$ and $\lambda m^2n(n-1)\equiv 0 \pmod{\beta(K)}$ together imply that $\lambda(n-1)\equiv 0 \pmod{\alpha(\Phi)}$ and $\lambda n(n-1)\equiv 0 \pmod{\beta(\Phi)}$.

To show $\lambda n(n-1)\equiv 0 \pmod{\beta(\Phi)}$, it suffices to show that the vector $\lambda n(n-1)(1,1,\ldots,1)$ is an integral linear combination of the vectors $\mu(G(\mathbf{t},k))$, $\mathbf{t}\in\mathcal{T}(m,k)$ and $k\in K$. The vectors $\mu(G(\mathbf{t},k))$ has m^2 coordinates indexed by the colors (i,j)'s, $i,j\in\{1,2,\ldots,m\}$; the coordinate at (i,i) is $t_i(t_i-1)$ and for $i\neq j$, the coordinate at (i,j) is t_it_j. By Lemma 2.2, to show a desired integral linear combination, it will suffice to show: whenever m^2 rational numbers x_{ij} are given, $1\leqslant i,j\leqslant m$, in such a way that

$$\sum_{i\neq j}t_it_jx_{ij}+\sum_i t_i(t_i-1)x_{ii}\equiv 0 \quad \text{for all } \mathbf{t}\in\mathcal{T}(m,k) \text{ and all } k\in K, \tag{2.1}$$

then

$$\lambda n(n-1)\sum_{i,j} x_{ij} \equiv 0,$$

where $a \equiv b$ means that the difference $a-b$ is an integer.

Assume (2.1) holds. For each $k \in K$ and each $1 \leqslant i \leqslant m$, fix $j \neq i$ and consider the three choices for $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ where $t_i = k$, where $t_i = k-1$, $t_j = 1$, and where $t_i = k-2$, $t_j = 2$ (all other coordinates being 0). By (2.1), we have

$$\begin{aligned} &k(k-1)x_{ii} \equiv 0, \\ &(k-1)(k-2)x_{ii} + (k-1)x_{ij} + (k-1)x_{ji} \equiv 0, \\ &(k-2)(k-3)x_{ii} + 2(k-2)x_{ij} + 2(k-2)x_{ji} + 2x_{jj} \equiv 0. \end{aligned} \tag{2.2}$$

Since the first congruence in (2.2) holds for all $k \in K$ and $\beta(K)$ is the greatest common divisor of the integers in $\{k(k-1)\colon k \in K\}$, it follows that $\beta(K)x_{ii} \equiv 0$, $1 \leqslant i \leqslant m$. If we add the first and the third equations and subtract twice the second in (2.2), we have

$$2x_{ij} + 2x_{ji} \equiv 2x_{ii} + 2x_{jj} \tag{2.3}$$

for any $i, j, i \neq j$. It implies that

$$n(n-1)x_{ij} + n(n-1)x_{ji} \equiv n(n-1)x_{ii} + n(n-1)x_{jj}$$

and thus

$$\lambda n(n-1)\sum_{i,j} x_{ij} \equiv \lambda n(n-1)m\sum_{i} x_{ii}. \tag{2.4}$$

If we subtract the second from the first in (2.2) we obtain

$$2(k-1)x_{ii} \equiv (k-1)(x_{ij} + x_{ji}) \tag{2.5}$$

and since this holds when i is replaced by j, we have

$$2(k-1)x_{ii} \equiv 2(k-1)x_{jj}$$

for all i and j and all $k \in K$. Since $\alpha(K)$ is the greatest common divisor of the integers in $\{k-1\colon k \in K\}$, it follows that

$$2\alpha(K)x_{ii} \equiv 2\alpha(K)x_{jj}$$

for all i and j. If $\alpha(K)$ is odd, since $\lambda m(n-1) \equiv 0 \pmod{\alpha(K)}$, we have $\lambda mn(n-1) \equiv 0 \pmod{2\alpha(K)}$, and so

$$\lambda mn(n-1)x_{ii} \equiv \lambda mn(n-1)x_{jj}. \tag{2.6}$$

If $\alpha(K)$ is even, then each $k \in K$ is odd, we multiply (2.3) by $\frac{k-1}{2}$ and combine it with (2.5) to obtain $(k-1)x_{ii} \equiv (k-1)x_{jj}$ for each $k \in K$. Thus, $\alpha(K)x_{ii} \equiv \alpha(K)x_{jj}$ and we again have (2.6). Since $\lambda m^2 n(n-1) \equiv 0 \pmod{\beta(K)}$ and $\beta(K)x_{ii} \equiv 0$ for each $1 \leqslant i \leqslant m$, it follows from (2.4) and (2.6) that

$$\lambda n(n-1)\sum_{i,j} x_{ij} \equiv \lambda n(n-1)m\sum_{i} x_{ii} \equiv \lambda m^2 n(n-1)x_{11} \equiv 0.$$

Thus, we have proved $\lambda n(n-1) \equiv 0 \pmod{\beta(\Phi)}$.

Now, we show that $\lambda(n-1) \equiv 0 \pmod{\alpha(\Phi)}$ assuming that $\lambda m(n-1) \equiv 0 \pmod{\alpha(K)}$. From earlier discussion, it suffices to show that the vector $\lambda(n-1)(1, 1, \ldots, 1)$ is an integral linear combination of the vectors $\mathbf{d}(x)$, as x ranges over all vertices of digraphs $G(\mathbf{t}, k)$ for all $\mathbf{t} \in \mathcal{T}(m, k)$ and $k \in K$.

A vector $\mathbf{d}(x)$ for a vertex x of $G(\mathbf{t}, k)$ has $2m^2$ coordinates, corresponding to the color (i, j) indegrees and the color (i, j) outdegrees. For $\mathbf{t} = (t_1, t_2, \dots, t_m)$ and $V(G(\mathbf{t}, k)) = T_1 \cup T_2 \cup \dots \cup T_m$ where $|T_i| = t_i$, if x is a vertex in T_q, then the color (i, q) indegree and the color (q, i) outdegree at x are t_i for $i \neq q$ and $t_q - 1$ for $i = q$, all other color (i, j) indegrees and color (i, j) outdegrees at x are zero.

By Lemma 2.2, to establish a desired integral linear combination, we need to show: Whenever $2m^2$ rational numbers x_{ij}, y_{ij} are given, $1 \leqslant i, j \leqslant m$, in such a way that

$$(t_q - 1)(x_{qq} + y_{qq}) + \sum_{i \neq q} t_i (x_{iq} + y_{qi}) \equiv 0$$
$$\text{for all } \mathbf{t} \in \mathcal{T}(m, k) \text{ and all } k \in K,\ 1 \leqslant q \leqslant m, \tag{2.7}$$

then

$$\lambda(n-1) \sum_{i,j} (x_{ij} + y_{ij}) \equiv 0.$$

Assume (2.7) holds. For each $k \in K$ and each $1 \leqslant q \leqslant m$, consider the choices for $\mathbf{t} = (t_1, t_2, \dots, t_m) \in \mathcal{T}(m, k)$ where $t_q = k$ and where $t_q = k - 1$ and $t_i = 1$, from (2.7) we have

$$(k-1)(x_{qq} + y_{qq}) \equiv 0, \tag{2.8}$$
$$(k-2)(x_{qq} + y_{qq}) + (x_{iq} + y_{qi}) \equiv 0. \tag{2.9}$$

Thus,

$$\alpha(K)(x_{qq} + y_{qq}) \equiv 0.$$

If we subtract (2.8) from (2.9), we obtain

$$(x_{iq} + y_{qi}) \equiv (x_{qq} + y_{qq}) \quad \text{for all } i \neq q.$$

Since $\lambda m(n-1) \equiv 0 \pmod{\alpha(K)}$ and $\alpha(K)(x_{qq} + y_{qq}) \equiv 0$ for each $1 \leqslant q \leqslant m$, it follows that

$$\lambda(n-1) \sum_{i,q} (x_{iq} + y_{qi}) \equiv \lambda m(n-1) \sum_{q} (x_{qq} + y_{qq}) \equiv 0.$$

Thus we have shown that $\lambda(n-1) \equiv 0 \pmod{\alpha(\Phi)}$.

Finally, we must show that Φ is admissible. From our earlier discussion, it suffices to show that there exists a positive rational linear relation

$$(1, 1, \dots, 1) = \sum_{G \in \Phi} c_G \mu(G) \quad \text{with all } c_G > 0.$$

Let $\mathbf{c}_1$ denote the sum of $\mu(G(\mathbf{t}, k))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length m with k in one coordinate and 0 elsewhere, for a fixed $k \in K$. Then $\mathbf{c}_1$ has coordinates s_{ij} where $s_{ii} = k(k-1)$ for all $1 \leqslant i \leqslant m$ and $s_{ij} = 0$ for $i \neq j$. Let $\mathbf{c}_2$ denote the sum of $\mu(G(\mathbf{t}, k))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length m that sum to k for every $k \in K$. Then $\mathbf{c}_2$ has coordinates u_{ij} such that $u_{ii} = a$ for all i and $u_{ij} = b$ for $i \neq j$, where a and b are constants. In fact, it is easy to see that if $\mathbf{c}(k)$ is the sum of $\mu(G(\mathbf{t}, k))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length m that sum to k for a fixed $k \in K$, then $\mathbf{c}(k)$ has coordinates c_{ij} where $c_{ii} = a_k$ for all i and $c_{ij} = b_k$ for $i \neq j$ with a_k and b_k being constants for a fixed k. Thus, for $a \leqslant b$, the linear combination $\frac{b-a}{k(k-1)}\mathbf{c}_1 + \mathbf{c}_2$ is a constant vector $(b, b, \dots, b)$, where $\frac{b-a}{k(k-1)} \geqslant 0$ and $b > 0$. For $a > b$, let $k \in K$ be fixed and let $\mathbf{c}_3$ be the sum of $\mu(G(\mathbf{t}, k))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length m that sum to k and have coordinates as equal as possible, that is, when we write $k = hm + p$ with $0 \leqslant p < m$, then $\mathbf{t}$ has $m - p$ coordinates equal to h and p coordinates equal to $h+1$. Then it is easy to check that $\mathbf{c}_3$ has coordinates h_{ij} such that for some constants c, d with $c < d$, $h_{ii} = c$ for all i and $h_{ij} = d$ for $i \neq j$. Thus, the linear combination $\frac{a-b}{d-c}\mathbf{c}_3 + \mathbf{c}_2$ produces a constant vector with each coordinate being $\frac{ad-bc}{d-c} > 0$, where $\frac{a-b}{d-c} > 0$. This completes the proof of the theorem. □

3. Asymptotic existence of frames

We first recall that a $\{k\}$-frame of type g^u is a group divisible design $\{k\}$-GDD of type g^u whose blocks are partitioned into partial parallel classes. The following GDD construction for $\{k\}$-frames is Corollary 2.4.3 with $\lambda = 1$ in [4].

Construction 3.1. Let K be a set of integers greater than 1 and $(X, \mathcal{G}, \mathcal{B})$ be a group divisible design with block sizes in K and $\lambda = 1$, and let $w(x)$ be a nonnegative integer-valued function on X. Suppose that for each $B \in \mathcal{B}$, there is a $\{k\}$-frame of type $\{w(x)\colon x \in B\}$. Then there is a $\{k\}$-frame of type $\{\sum_{x \in G} w(x)\colon G \in \mathcal{G}\}$.

Next we give a simple lemma.

Lemma 3.2. *For any integers $d \geqslant 1$ and $k \geqslant 2$, let $a = (k-1)(k+2)^d$. Then* $\gcd((ak+1)a, [(a+1)k+1](a+1)) = 1$ *if k is even and* $\gcd((ak+1)a, [(a+1)k+1](a+1)) = 2$ *if k is odd.*

Proof. Clearly, $\gcd(a, a+1) = 1$. As $(a+1)k + 1 = ak + 1 + k$, $\gcd(ak+1, (a+1)k+1) = \gcd(ak+1, k) = 1$. Since $k-1 = (a+1)k - (ak+1)$ and $a = (k-1)(k+2)^d$, we have $\gcd(a+1, ak+1) = \gcd(a+1, k-1) = 1$. To prove the lemma, it remains to show that $\gcd(a, (a+1)k + 1) = 1$ if k is even and $\gcd(a, (a+1)k+1) = 2$ if k is odd. Since $(a+1)k + 1 = ak + k + 1$, $\gcd(a, (a+1)k+1) = \gcd(a, k+1)$. By the formula for the sum of a geometric sequence, we have

$$1 + (k+2) + (k+2)^2 + \cdots + (k+2)^{d-1} = \frac{(k+2)^d - 1}{(k+2) - 1}.$$

It follows that

$$(k+2)^d = (k+1)\big[1 + (k+2) + (k+2)^2 + \cdots + (k+2)^{d-1}\big] + 1$$

and $\gcd((k+2)^d, k+1) = 1$. Since $a = (k-1)(k+2)^d$, we conclude that $\gcd(a, k+1) = \gcd(k-1, k+1)$. Clearly, $\gcd(k-1, k+1)$ divides $(k+1) - (k-1) = 2$. Thus, we have $\gcd(a, (a+1)k+1) = \gcd(a, k+1) = \gcd(k-1, k+1) = 1$ or 2. For k even, both $k-1$ and $k+1$ are odd, so we have $\gcd(k-1, k+1) = 1$. For k odd, then both $k-1$ and $k+1$ are even, thus, we have $\gcd(k-1, k+1) = 2$. Thus, we have shown that $\gcd(a, (a+1)k+1) = 1$ if k is even and $\gcd(a, (a+1)k+1) = 2$ if k is odd, and so the lemma follows. □

Proof of Theorem 1.9. Let $g = (k-1)m$. Then $g(u-1) \equiv 0 \pmod{k}$ implies that $m(u-1) \equiv 0 \pmod{k}$. First, we claim that a $\{k\}$-frame of type $(k-1)^h$ exists for h sufficiently large and $h - 1 \equiv 0 \pmod{k}$. In fact, let $v = (k-1)h + 1$, then $v - 1 \equiv 0 \pmod{k-1}$ and $v \equiv 0 \pmod{k}$.

Thus, by Theorem 1.4, there exists v_0 such that for $v \geqslant v_0$, a resolvable $(v, k, 1)$-design exists. By deleting one vertex x and all blocks containing x, we obtain a $\{k\}$-frame of type $(k-1)^h$ for $h \geqslant h_0$, where h_0 is some constant.

Now suppose that $a = (k-1)(k+2)^d$ is a constant with d sufficiently large so that $ak + 1 \geqslant h_0$. Let $K = \{ak+1, (a+1)k+1\}$. Clearly $\gcd(ak, (a+1)k) = k$. By Lemma 3.2, $\gcd((ak+1)ak, [(a+1)k+1](a+1)k) = k$ if k is even and $\gcd((ak+1)ak, [(a+1)k+1](a+1)k) = 2k$ if k is odd, which implies that $\alpha(K) = k$, $\beta(K) = k$ for k even and $\beta(K) = 2k$ for k odd. Since $m(u-1) \equiv 0 \pmod{k}$, we have $m(u-1) \equiv 0 \pmod{\alpha(K)}$. We claim that $m^2u(u-1) \equiv 0 \pmod{\beta(K)}$. In fact, the claim is obvious for k even as $\beta(K) = k$ in this case. For k odd, since $u(u-1)$ is even, $m(u-1) \equiv 0 \pmod{k}$, and $\beta(K) = 2k$, we also have $m^2u(u-1) \equiv 0 \pmod{\beta(K)}$. Thus, the claim holds. By Theorem 1.8, there exists u_0 such that a group divisible design $(\{ak+1, (a+1)k+1\}, 1)$-GDD of type m^u exists for $u \geqslant u_0$.

Since $ak + 1 \geqslant h_0$ and $(a+1)k + 1 \geqslant h_0$, a $\{k\}$-frame of type $(k-1)^{ak+1}$ and a $\{k\}$-frame of type $(k-1)^{(a+1)k+1}$ exist. By applying Construction 3.1 with $w(x) = k-1$ for every $x \in X$, $|\mathcal{G}| = u$, and each group having size m, we obtain a $\{k\}$-frame of type g^u, where $g = (k-1)m$ and $u \geqslant u_0$. □

4. Resolvable group divisible designs

A *transversal design TD*(k, m) is defined to be a $\{k\}$-GDD of type m^k, where the number of groups is the same as the size k of blocks, i.e., each block takes exactly one element from every group. The following result is well known [1].

Proposition 4.1. *A resolvable TD*(k, m) *exists if and only if there are* $k-1$ *mutually orthogonal Latin squares of order* m.

It was shown by Chowla, Erdős, and Straus [3] that the number of mutually orthogonal Latin squares of order m approaches infinity as m goes to infinity. Thus, we have the next lemma.

Lemma 4.2. *Given a fixed integer* $k \geqslant 2$, *there exists* m_0 *such that a resolvable TD*(k, m) *exists for all* $m \geqslant m_0$.

A *factor* F of a graph G is a subgraph of G for which $V(F) = V(G)$. Let $K(m:n) = K(m, m, \ldots, m)$ denote a complete n-partite graph $K(m, m, \ldots, m)$ with m vertices in each partite set. A K_k*-factorization* of a graph G is a partition of the edge set $E(G)$ into isomorphic factors where each factor is a disjoint union of K_k's. Then, by viewing each block of size k as a complete graph K_k, it is easy to see that a resolvable group divisible design $\{k\}$-RGDD of type m^n is a K_k-factorization of $K(m:n)$. Thus, we have the following well-known necessary conditions for the existence of resolvable group divisible designs.

Proposition 4.3. *The necessary conditions for the existence of a* $\{k\}$*-RGDD of type* m^n *are* $m(n-1) \equiv 0 \pmod{k-1}$ *and* $mn \equiv 0 \pmod{k}$.

Here we offer the following asymptotic existence conjecture for resolvable group divisible designs.

Conjecture 4.4. *Given integers $k \geqslant 2$ and $m \geqslant 1$, there exists n_0 such that a $\{k\}$-RGDD of type m^n exists for all integers $n \geqslant n_0$ that satisfy the necessary conditions $m(n-1) \equiv 0 \pmod{k-1}$ and $mn \equiv 0 \pmod{k}$.*

Recall that a $\{k\}$-frame of type g^u is a group divisible design $\{k\}$-GDD of type g^u whose blocks are partitioned into partial parallel classes, or equivalently, it is a K_k-decomposition of $K(g:u)$ such that the subgraphs K_k's are partitioned into partial parallel classes where each partial parallel class forms a factor of $K(g:u-1)$ (a subgraph of $K(g:u)$ after removing one group of g vertices). By a simple calculation, it follows that a $\{k\}$-frame of type g^u has $\frac{gu}{k-1}$ partial parallel classes in total and has exactly $\frac{g}{k-1}$ partial parallel classes excluding each group G_i (called a hole).

Next, we provide a simple but useful recursive construction for resolvable group divisible designs.

Construction 4.5 *(Filling in holes).* Let m and g be positive integers such that g is divisible by m. Suppose that there exists a $\{k\}$-frame of type g^u and there exists a $\{k\}$-RGDD of type $m^{\frac{g+m}{m}}$. Then there exists a $\{k\}$-RGDD of type m^n with $n = \frac{g}{m}u + 1$.

Proof. Start with a $\{k\}$-frame of type g^u and let W be a set of m elements not from the frame. For each group G_i of size g in the frame, we fill the hole G_i by a $\{k\}$-RGDD of type $m^{\frac{g+m}{m}}$ on the set $G_i \cup W$, i.e., match the parallel classes of a $\{k\}$-RGDD of type $m^{\frac{g+m}{m}}$ with the partial parallel classes excluding G_i of the $\{k\}$-frame of type g^u to form parallel classes of the whole design. □

Here is another construction method which is a special case of Corollary 3.5.5 with $\lambda = 1$ in [4].

Construction 4.6. Suppose that the following designs exist:

(1) a $\{k\}$-RGDD of type g^u,
(2) a $\{k\}$-frame of type $(m_1 g)^v$,
(3) a resolvable $TD(k, m_1 v)$.

Then there exists a resovable $\{k\}$-RGDD of type $(m_1 g)^{uv}$.

The following results provide a partial solution to Conjecture 4.4.

Theorem 4.7. *Given an integer $k \geqslant 2$, there exist m_0 and n_0 such that a $\{k\}$-RGDD of type m^n exists for all integers $m \geqslant m_0$ and $n \geqslant n_0$ that satisfy $(n-1) \equiv 0 \pmod{k-1}$ and $mn \equiv 0 \pmod{k}$.*

Proof. Let $g = (k-1)m$ and $u = \frac{n-1}{k-1}$. Then $g \equiv 0 \pmod{k-1}$ and $\frac{g+m}{m} = k$, and so $g(u-1) \equiv 0 \pmod{k}$. By Theorem 1.9, there exists u_0 such that a $\{k\}$-frame of type g^u exists for $u \geqslant u_0$. Recall that a resolvable $TD(k, m)$ is a $\{k\}$-RGDD of type m^k. By Lemma 4.2, a $\{k\}$-RGDD of type m^k exists for $m \geqslant m_0$, where m_0 is some constant. Since $k = \frac{g+m}{m}$, it follows from Construction 4.5 that a $\{k\}$-RGDD of type m^n exists, where $n = (k-1)u + 1 = \frac{g}{m}u + 1$. □

Theorem 4.8. *Given an integer $k \geqslant 2$, there exist m_0 and n_0 such that a $\{k\}$-RGDD of type m^n exists for all integers $m \geqslant m_0$ and $n \geqslant n_0$ that satisfy one of the following*:

(1) $m \equiv 0 \pmod{k(k-1)}$ *and* $n \equiv 0 \pmod{k}$, *or*
(2) $m \equiv 0 \pmod{(k-1)}$ *and* $n \equiv 0 \pmod{k^2}$.

Proof. We first prove the result for condition (1). Set $m = (k-1)g$ and $n = kv$. Since k divides m, it follows from Theorem 1.9 that a $\{k\}$-frame of type $[(k-1)g]^v$ exists for $v \geqslant v_0$, namely, $n = kv \geqslant n_0$ for some n_0. By Lemma 4.2, resolvable $TD(k, g)$ and resolvable $TD(k, (k-1)g)$ exist for all $g \geqslant g_0$, namely, $m = (k-1)g \geqslant m_0$ for some m_0. Recall that a resolvable $TD(k, g)$ is a $\{k\}$-RGDD of type m^k. By applying Construction 4.6 with $u = k$ and $m_1 = k-1$, we obtain a $\{k\}$-RGDD of type m^n.

To prove the result for condition (2), let $n_1 = \frac{n}{k}$. Then $n_1 \equiv 0 \pmod{k}$. It is easy to see that the complete n-partite graph $K(m:n)$ is a disjoint union of the factors $H = \bigcup K(m:k)$ and $K(mk:n_1)$. By Lemma 4.2, a resolvable $TD(k, m)$ exists for $m \geqslant m_0$, i.e., a $\{k\}$-RGDD of type m^k exists which means that $K(m:k)$ has a K_k-factorization, and so is $H = \bigcup K(m:k)$. By (1), a $\{k\}$-RGDD of type $(mk)^{n_1}$ exists which means that $K(mk:n_1)$ has a K_k-factorization. Thus, $K(m:n) = H \cup K(mk:n_1)$ has a K_k-factorization, that is, a $\{k\}$-RGDD of type m^n. □

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